Nonlinear Structural Dynamic Analysis Using a Modified Modal Method

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A procedure for predicting the nonlinear dynamic response of structural components subjected to a step loading is presented. The procedure is a modified modal method that involves a change of dependent variables from the unknown nodal degrees of freedom of the finite element model of the structure to a smaller set of generalized coordinates. This change of dependent variables uses a combination of the nonlinear static solution and some selected vibration mode shapes. The vibration mode shapes correspond to the eigenvectors obtained by solving a standard free vibration eigenvalue problem wherein the stiffness matrix is expanded about the nonlinear static solution. A strategy is also presented for determining which and how many vibration mode shapes to include in the transformation. The effect of inaccurate representation of the spatial distribution of the applied load on the nonlinear dynamic response is discussed for two classes of structural behavior. Application of the procedure to structures that exhibit a stiffening behavior and to those with a softening behavior is presented.

Nomenclature

a	= half-length of side of plate or shell structure
E	= elastic modulus
$[F]$	= linear flexibility matrix of the full system of equations
$[f]$	= linear flexibility matrix of reduced system of equations, see Eq. (13a)
$\{G(H,X)\}_t$	= vector of nonlinear coefficients; bilinear in $\{H\}_t$ and $\{X\}_t$
$\{g(X)\}_t$	= vector of nonlinear coefficients, quadratic in $\{X\}_t$
H	= maximum rise of shell
$\{H_S\}$	= vector of stress-resultant parameters of static solution
$\{H\}_t$	= vector of unknown stress-resultant parameters at time t
h	= thickness of plate or shell
$\{h\}_t$	= vector of generalized coordinates for stress-resultant parameters of reduced system of equations at time t
L	= number of displacement degrees of freedom of full system of equations
ℓ	= number of displacement generalized coordinates or reduced basis vectors
$[M]$	= mass matrix of full system of equations
$[m]$	= mass matrix of reduced system of equations, see Eq. (13c)
N	= number of stress-resultant degrees of freedom of full system of equations
n	= number of stress-resultant generalized coordinates or reduced basis vectors
$\{P\}$	= consistent nodal vector of full system of equations
$\{p\}$	= load vector of the reduced system of equations, see Eq. (13d)
$p_0(t)$	= temporal variation of load
R	= radius of curvature of shell structure
q_0	= classical buckling pressure of a complete spherical shell, see Eq. (14)
$[S]$	= linear generalized stiffness matrix of full system of equations
$[s]$	= linear generalized stiffness matrix of reduced system of equations, see Eq. (13b)
$[T_u^\infty]$	= matrix of vibration mode shapes
t	= time
Δt	= size of time step
U	= strain energy
$\{u\}_t$	= vector of generalized coordinates for the displacements of reduced system of equations at time t
$\{\dot{u}\}_t$	= vector of generalized coordinates for the velocities of reduced system of equations at time t
$\{X\}_S$	= vector of displacements of the static solution
$\{X\}_t$	= vector of unknown displacements at time t
$\{\dot{X}\}_t$	= vector of unknown velocities at time t
$\{Y_H\}_i$	= ith stress-resultant mode shape corresponding to $\{Y_X\}_i$, see Eq. (8)
$\{Y_X\}_i$	= ith vibration displacement mode shape, see Eq. (7)
α_i	= participation coefficient of the ith vibration mode shape, see Eq. (9a)
$\bar{\alpha}_i$	= ith normalized participation coefficient, see Eq. (9b)
$[\Gamma_h]$	= transformation matrix for stress resultants, see Eq. (6)
$[\Gamma_u]$	= transformation matrix for diplacements, see Eq. (5)
λ	= shell shallowness parameter, $2[3(1-\nu^2)]^{1/4}\sqrt{H/h}$
ν	= Poisson's ratio
ρ	= mass density of the material

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Introduction

THE nonlinear dynamic response of a structure can be predicted by the numerical integration of the equations of motion that result from a spatial finite element discretization of the structure. The cost of the dynamic analysis is governed primarily by the form of nonlinearity, the variation of the applied load with time, the number of degrees of freedom in the discretized model of the structure, and the size of the time step for a given time integrator. The analyst usually has little control over the form of nonlinearity or the variation of the applied load with time for the structure under consideration. However, some control of the analysis cost is permitted in the selection of the time integration algorithm (e.g., explicit or implicit) and the finite element discretization of the structure. The cost of a nonlinear dynamic analysis of a finite element model of a realistic structure, involving hundreds or even thousands of degrees of freedom, may be prohibitive unless a reduction method is used. Reduction methods transform the unknown nodal degrees of freedom of the full system of equations to a smaller set of unknown parameters or generalized coordinates of the reduced system of equations. The link between the full and reduced systems of equations is a transformation matrix. The columns of this transformation matrix are referred to as reduced basis vectors. The reduction methods that have been developed to date for the dynamic analysis of structures may be classified as either Taylor series expansion methods or modal superposition methods. The reduction methods combine the use of contemporary finite elements and the classical Rayleigh-Ritz technique, thereby preserving the modeling versatility of the finite element method and simultaneously reducing the number of degrees of freedom through the Rayleigh-Ritz approximations.

Modal superposition methods are perhaps the most popular class of reduction methods for the dynamic analysis of structures. Application of modal superposition of linear dynamics problems is commonplace in the engineering community. The use of a modal transformation matrix gives a set of uncoupled linear equations of motion. The resulting independent equations may then be integrated separately and superimposed until a converged solution is obtained. The number of mode shapes necessary for accurate response determination continues to be the main difficulty associated with applying this approach to dynamic analysis. Application of modal superposition to nonlinear dynamics problems generally takes one of two forms, depending on how the nonlinear terms are treated. Horii and Kawahara[1] and Nickell[2] have suggested solving the free vibration eigenvalue problem at various times during the dynamic analysis. This approach of updating the modal transformation matrix with the instantaneous free vibration mode shapes accounts for changes in structural stiffness as a function of time. However, repeated solution of the free vibration eigenvalue problem is required throughout the transient response prediction. Wilke,[3] Molnar et al.,[4] and Bathe and Gracewski[5] have suggested using the initial free vibration mode shapes throughout the dynamic analysis, provided the nonlinearities of the problem are transferred to the right-hand side of the equations of motion. Treating the nonlinear terms in this manner is known as a pseudoforce approach. However, each time the pseudoforces are updated, two additional matrix multiplications are required: one to obtain the nodal displacements of the full system of equations from the generalized coordinates of the reduced system of equations, and one to transform the pseudoforce vector from the full system back to the reduced system. These additional matrix manipulations can add substantially to the cost of the analysis for large problems.

Another modal superposition method has been presented by Noor,[6] which incorporates ideas from both the traditional modal superposition approach used in linear dynamic analyses and updated or instantaneous modal superposition method used in nonlinear dynamic analyses. In contrast to the previous modal methods, Noor has suggested that a combination of two sets of vibration mode shapes be used as the reduced basis vectors. The first set of vibration mode shapes corresponds to the eigenvectors from the standard free vibration eigenvalue problem expanded about the unloaded equilibrium configuration. The second set of vibration mode shapes corresponds to the eigenvectors from a free vibration eigenvalue problem expanded about the nonlinear static solution for a load intensity of the step loading case. This set of reduced basis vectors requires the solution of two free vibration eigenvalue problems.

To date, the application of reduction methods to problems in nonlinear structural dynamics has been limited. The objectives of this paper are to present a reduction method for the dynamic analysis of two-dimensional, geometrically nonlinear structures subjected to step loadings and to present a strategy for selecting the reduced basis vectors. The reduced basis vectors of the present reduction method include the nonlinear static solution corresponding to the load intensity of the step loading case and the free vibration mode shapes about the nonlinear static equilibrium configuration. As such, the present method is referred to as a modified modal method. The strategy for selecting the reduced basis vectors includes criteria for determining the number of vibration mode shapes to calculate and which of the calculated mode shapes to include as reduced basis vectors. Numerical results are presented for transient responses typical of plate and shell structures subjected to step pressure loadings.

Mathematical Formulation

The finite elements used to model the structures in this study are shear-flexible, shallow-shell finite elements based on a modified form of the Hellinger-Reissner mixed variational principle[7,8] and a geometrically nonlinear shallow-shell theory with the effects to transverse shear, bending-extensional coupling, and rotary inertia included.[9] Since the fundamental nodal unknowns for these elements include 5 generalized displacements, 5 generalized velocities, and 8 stress resultants, the finite element model is referred to as a mixed model. The advantages of mixed models compared to displacement models have been shown by Noor and Peters[9] to include the simplicity of development of the elemental matrices, the uniform accuracy and convergence properties for a wide range of geometric characterisitics of the shell, and the accuracy and availability of the stress resultants at each step in the analysis. A total Lagrangian formulation is used, wherein the deformations are referenced to the original undeformed geometry of the structure. The effects of damping are neglected in this paper. In the following sections, the full system of equations of motion for a mixed finite element model are developed and the method of time integration used in this study is described.

Full System of Equations of Motion

The equations of motion for each element are obtained by first replacing the stress resultants, generalized displacements, and velocity components that occur in the functional from the mixed variational principle by their expressions in terms of the element shape functions. Next, the stationary condition is applied to the functional from the mixed variational principle. If the nodal parameters are varied independently and simultaneously, and the variations of the quantities are assumed to vanish at the beginning and end of the time interval of interest, a mixed set of algebraic and ordinary differential equations for each element are obtained. This mixed set of equations is referred to as a set of semidiscrete equations (i.e., algebraic equations spatially and ordinary differential equations temporally). The full system of equations for the mixed model is obtained by combining the nodal contributions from the different elements. The semidiscrete equations of motion of the mixed model are as follows:

$$[F]\{H\}_t = [S]\{X\}_t + \{g(X)\}_t \tag{1}$$

$$[M]\frac{\mathrm{d}}{\mathrm{d}t}\{\bar{X}\}_t = p_0(t)\{P\} - [S]^T\{H\}_t - \{G(H,X\}_t \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{X\}_t = \{\bar{X}\}_t \tag{3}$$

where $\{X\}_t$, $\{\dot{X}\}_t$, and $\{H\}_t$ are nodal displacements, velocities, and stress-resultant parameters, respectively, at time t; $[F]$ represents a linear flexibility matrix, $[S]$ a generalized linear stiffness matrix, $[M]$ a mass matrix, and $\{P\}$ a consistent nodal load vector; $\{g(X)\}_t$ and $\{G(H,X)\}_t$ represent vectors of nonlinear terms in the equations, and the superscript T denotes transposition. The matrices $[F]$ and $[M]$ are symmetric, positive definite matrices that may be banded and the matrix $[S]$ is sparse. The vectors $\{g(X)\}_t$ and $\{G(H,X)\}_t$ are a result of the nonlinear terms in the strain-displacement relations of the shell. The variation of the loading intensity with time is given by the function $p_0(t)$, which is assumed to be a constant in this study. Equations (1-3) are referred to herein as the full system of equations.

Method of Time Integration

With the specification of the initial conditions, the full system of mixed algebraic and ordinary differential equations, Eqs. (1) to (3), can be solved to obtain the transient response of the structure. The time integration algorithm used in this study was the half-station central difference operator. This operator requires the derivative with respect to time t of a function $f(t)$ to be approximated by using values of the function at half-stations in time (i.e., at $t \pm \Delta t/2$). Thus, the half-station central difference operator may be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) = \frac{1}{\Delta t}\left[f\left(t+\frac{\Delta t}{2}\right) - f\left(t-\frac{\Delta t}{2}\right)\right] \tag{4}$$

Using this operator with a mixed finite element model requires the generalized velocities to be evaluated at points lying midway, in the time domain, between points of definition of the stress-resultant parameters and the generalized displacements.

Since the half-station central difference operator is only conditionally stable, the size of the time step Δt must be smaller than the size of the critical time step, which is calculated from the highest modeled frequency of the structure. Although strictly applicable to linear problems, this numerical stability limit does provide useful estimates of the size of the time step for nonlinear problems. Experience has shown that the size of the largest allowable time step that can be used with an explicit time integrator and still maintain numerically stability for nonlinear systems is approximately 90% of the critical time step.

Reduction Method for Nonlinear Dynamics

A reduction method transforms the unknown nodal parameters of the full system of equations to the generalized coordinates of the reduced system of equations. The link between the full and reduced systems is a transformation matrix. The columns of this transformation matrix are referred to as reduced basis vectors. For a mixed finite element model, two transformation matrices are required. One transformation matrix is needed for the generalized displacements

$$\{X\}_{L,1} = [\Gamma_u]_{L,\ell}\{u\}_{\ell,1} \qquad L \gg \ell \tag{5}$$

and another for the stress-resultant parameters

$$\{H\}_{N,1} = [\Gamma_h]_{N,n}\{h\}_{n,1} \qquad N \gg n \tag{6}$$

The number of nodal degrees of freedom of the full system of equations for the stress-resultant parameters and generalized displacements are denoted by N and L, respectively. The number of generalized coordinates or reduced basis vectors of the reduced system of equations for the stress-resultant parameters and generalized displacements are denoted by n and ℓ, respectively.

The use of a reduction method gives rise to a least three additional questions that relate to the solution accuracy and cost. The analyst must first, select the type of reduced basis vectors to use in the transformation matrix; second, determine how many reduced basis vectors are required; and third, decide how often the accuracy of the reduced solution should be verified.

Selection of the Reduced Basis Vectors

Selection of an appropriate set of reduced basis vectors is one of the key elements in a successful reduced system transient analysis. The analyst must decide on the type and number of reduced basis vectors to include in the transformation matrix that links the full system of equations to the reduced system of equations. The reduction method developed herein represents an extension of the traditional modal superposition approach used in linear dynamics analyses and the updated modal superposition approach used in nonlinear dynamics analyses. In contrast to the traditional and updated modal superposition methods, the present method uses a combination of two types of reduced basis vectors. The generalized displacements of the full system of equations are transformed to the corresponding generalized coordinates of the reduced system of equations using the following equation

$$\{X\}_t = \sum_{i=1}^{\ell-1}\{Y_X\}_i u_i(t) + \{X_S\}u_\ell(t) \tag{7a}$$

or

$$\{X\}_t = \left[[T_u^\infty] \;\vdots\; \{X_S\}\right]\{u\}_t \tag{7b}$$

The first type of reduced basis vectors in Eq. (7b) is a set of free vibration mode shapes of the structure $[T_u^\infty]$. These mode shapes correspond to the eigenvectors associated with a standard eigenvalue problem wherein the stiffness matrix is expanded about the nonlinear static solution corresponding to the load intensity of the step loading case. The selection of these vibration mode shapes is a natural choice since the solution to an undamped initial-value problem corresponds to a steady-state vibration solution about the static equilibrium configuration. These vibration mode shapes will be referred to as the steady-state mode shapes.

The second type of reduced basis vector in Eq. (7b) is the nonlinear static solution for the specified step loading under consideration, $\{X_S\}$. Using the static solution as a reduced basis vector allows the corresponding generalized coordinate to vary with time and hence represent the time-averaged steady-state portion of the transient response. The use of the static solution as a reduced basis vector is conceptually similar to a Williams-type[10] modal method for the dynamic analysis of linear elastic systems. In a Williams-type modal solution, the transient response is separated into a static portion and a dynamic portion. Once the static portion has been isolated, the remaining dynamic portion may be solved using a variety of techniques as shown by Leonard.[11] The selection of the static solution is also a natural choice since the solution to a damped initial-value problem will eventually reach the static solution.

The set of reduced basis vectors for the stress-resultant parameters are the corresponding stress-resultant mode shapes obtained from Eq. (5) and the appropriate portion of the static solution $\{H_S\}$. That is, the stress-resultant parameters of the full system of equations are transformed to the corresponding generalized coordinates of the reduced system of equations using the following equation:

$$\{H\}_t = \sum_{i=1}^{n-1}\{Y_H\}_i h_i(t) + \{H_S\}h_n(t) \tag{8}$$

The number of vibration mode shapes to include as reduced basis vectors may be determined using two criteria. The first criterion involves the number of vibration mode shapes that can be represented accurately for a given finite element discretization of the structure being modeled. For two-dimensional structures, an accurate representation of a vibration mode shape requires three to five nodes in each coordinate direction per half-wave of the mode shape. This criterion should be invoked by the analyst when specifying the number of eigenvectors or vibration mode shapes to be calculated. The second criterion involves the orthogonality of the spatial distribution of the applied load and the vibration mode shapes. The orthogonality criterion provides a measure $\bar{\alpha}_i$ of the anticipated participation of each mode shape $\{Y_X\}_i$ in the total solution. This relationship may be expressed as

$$\alpha_i = \frac{1}{\omega_i^2} \left| \{Y_X\}_i^T \{P\} \right| \tag{9a}$$

$$\bar{\alpha}_i = \alpha_i / [\text{maximum}(\alpha_i)] \tag{9b}$$

The value of the normalized participation coefficients $\bar{\alpha}_i$ for the ith mode shape ranges between zero and one and must be greater than a prescribed tolerance before the ith mode shape is included as a reduced basis vector. A prescribed tolerance of zero will result in all of the calculated vibration mode shapes being included as reduced basis vectors. As the value of the precribed tolerance is increased above zero, fewer calculated vibration mode shapes are included. For example, if twenty vibration mode shapes were calculated for a given problem, then perhaps only the first, second, fifth, eleventh, and sixteenth mode shapes give a value of $\bar{\alpha}_i$ $(i=1,2,5,11,16)$ greater than a prescribed tolerance of 0.005. The orthogonality criterion given by Eq. (9) is used to select which of the calculated vibration mode shapes to include as reduced basis vectors.

In this study, the first criterion is used to specify the number of vibration mode shapes to calculate for a given finite element model of the structure. The second criterion is used to determine which of the calculated vibration mode shapes to include as reduced basis vectors.

Reduced System of Equations of Motion

The reduced system of equations of motion is obtained by substituting Eqs. (5) and (6) into Eqs. (1-3) and then premultiplying Eq. (1) by $[\Gamma_h]^T$ and Eq. (2) by $[\Gamma_u]^T$. The resulting reduced system of equations are as follows:

$$[f]\{h\}_t = [s]\{u\}_t + [\Gamma_h]^T\{g(u)\}_t \tag{10}$$

$$[m]\frac{\mathrm{d}}{\mathrm{d}t}\{\dot{u}\}_t = p_0(t)\{p\} - [s]^T\{h\}_t - [\Gamma_u]^T\{G(h,u)\}_t \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{u\}_t = \{\dot{u}\}_t \tag{12}$$

where

$$[f] = [\Gamma_h]^T[F][\Gamma_h] \tag{13a}$$

$$[s] = [\Gamma_h]^T[S][\Gamma_u] \tag{13b}$$

$$[m] = [\Gamma_u]^T[M][\Gamma_u] \tag{13c}$$

$$\{p\} = [\Gamma_u]^T\{P\} \tag{13d}$$

The initial conditions of the full system of equations are also transformed to the reduced system of equations. The reduced system transient response of the structure is then obtained by numerically integrating Eqs. (10-12), using the explicit half-station central difference operator. One important advantage of a reduction method based on a truncated modal expansion is that the size of the critical time step for the reduced system of equations depends only on the corresponding vibration frequencies of the mode shapes included as reduced basis vectors. However, the size of the critical time step for the full system of equations using an explicit time integrator depends on the highest modeled vibration frequency of the structure, which is rarely calculated. Thus, the size of the critical time step for the reduced system of equations is readily obtained and may be as much as an order of magnitude larger than that of the full system of equations. This increase in the size of the critical time step for modal methods is analogous to using an implicit time integrator on the full system of equations of motion, since the number of vibration mode shapes that will be integrated accurately depends on the size of the time step.

Numerical Studies

To demonstrate the accuracy of the present reduction method for the dynamic analysis of two-dimensional, geometrically nonlinear structures under a step loading condition, several examples are presented below. The example problems considered involve two classes of generic nonlinear structural response to lateral loads. Based on results from nonlinear dynamic analyses of flat plates and doubly-curved shells, the accuracy and limitations of the present reduction method are discussed.

The transient responses obtained by using the full system of equations, Eqs. (1) to (3), are compared with those obtained by using the reduced system of equations, Eqs. (10-12). The results obtained by the present reduction method are also compared with those obtained using the traditional linear modal superposition approach to demonstrate the accuracy of the present method.

For the present method, the vibration mode shapes corresponding to the 20 lowest vibration frequencies were calculated for each example problem. Then the orthogonality criterion was applied to each of the calculated mode shapes. The vibration mode shapes with a normalized participation coefficient $\bar{\alpha}_i > 0.005$ were included as reduced basis vectors.

Generic Nonlinear Structural Response

The nonlinear static response of a structure generally falls into one of two categories. If the stiffness of the structure as measured by the slope of the load-deflection curve increases for increasing values of load, the response is referred to as one with a stiffening behavior characteristic of a flat plate subjected to a lateral load. If the stiffness of the structure decreases for increasing values of load, the response is referred to as one with a softening behavior characteristic of a shell subjected to an external pressure load. This class of structures may exhibit a limit point in the nonlinear static response.

For the class of structures with a stiffening behavior, small variations of the lateral load will only produce small variations in the overall equilibrium configuration of the structure. A structure with a softening behavior may experience a sudden and often violent change in its overall equilibrium configuration if loaded above the critical value of the load. Dynamically, structures of this class may change equilibrium configurations continuously throughout the dynamic response. This class of structures presents a difficult challenge to any reduction method since the overall displacement field may change radically as the load level is increased or as the length of the time interval for the dynamic analysis is increased. The continual change of the overall displacement field is in contrast to the class of structures with a stiffening behavior that maintains essentially the same overall displacement field throughout the dynamic response. The effect of modal truncation on the spatial representation of the applied load and internal force distribution is also examined to define the limitations of the modified modal method.

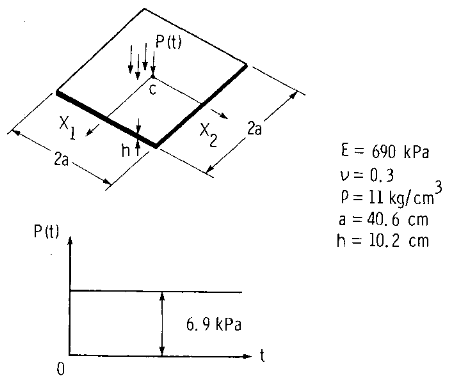


Fig. 1 Simply supported square plate and loading used.

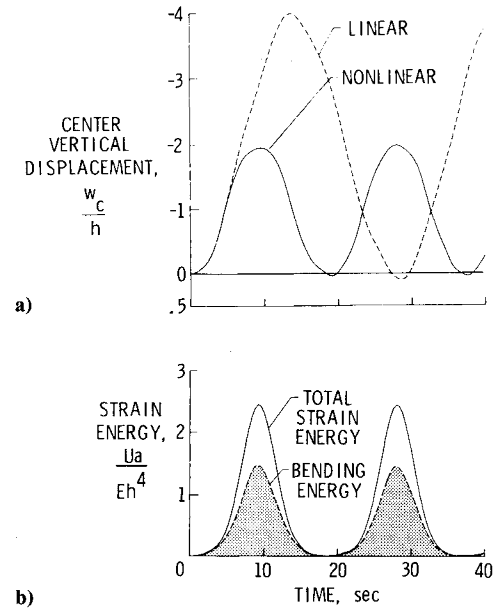


Fig. 2 Full system dynamic response of a simply supported square plate subjected to a step pressure load of 6.9 kPa ($\Delta t = 0.08$ s).

A simply supported square plate and two clamped shallow spherical shell segments subjected to a uniformly distributed lateral load are considered to demonstrate the accuracy and to define the limitations of the modified modal method. The simply supported square plate is an example to a flat structure that exhibits a stiffening behavior. A clamped shallow spherical shell segment with a shell shallowness parameter $\lambda = 2$ is an example of a structure with softening behavior that exhibits a monotonic postbuckling response for increasing values of the external pressure intensity.[12] A clamped shallow spherical shell segment with $\lambda = 5$ is an example of a structure with a softening behavior that exhibits a limit point in the postbuckling response. Numerical results selected from those of Ref. 13 are presented in this paper.

Simply Supported Square Plate

The behavior of the simply supported square plate shown in Fig. 1 has been analyzed for the case of a uniformly distributed lateral load. A quarter-model of the plate consisting of nine 9-node mixed elements was used for this study. A total of 288 stress-resultant degrees of freedom and 180 displacement degrees of freedom are considered as active or nonzero.

The dynamic response of the simply supported square plate subjected to a step load intensity of 6.9 kPa was also determined. The linear and nonlinear transient response of the plate predicted by the full system of equations are shown in Fig. 2. The nonlinear response shown in Fig. 2 indicates that dynamically the square plate is also stiffer than the linear solution predicts. The addition of the nonlinear effects not only makes the square plate stiffer but also decreases the response period as shown by the temporal variation of the center vertical displacement in Fig. 2a. Near peaks of the center vertical displacement, a large increase in the ratio of extensional to bending strain energies from the nonlinear solution occurs as shown in Fig. 2b. This ratio of strain energy components indicates the relative participation of the extensional and bending modes of deformation to the overall response of the structure.

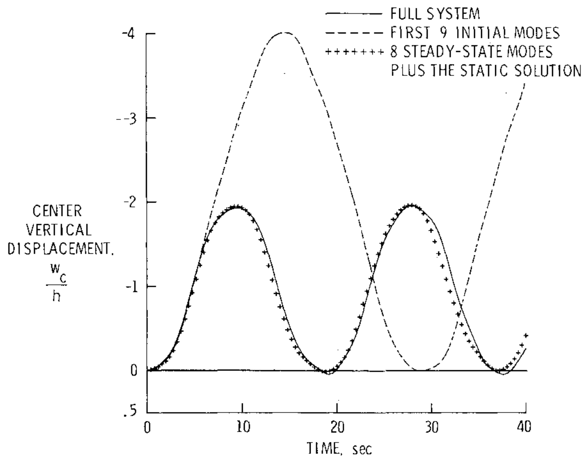


Fig. 3 Reduced system nonlinear dynamic response of a simply supported square plate subjected to a step pressure load of 6.9 kPa ($\Delta t = 0.5$ s).

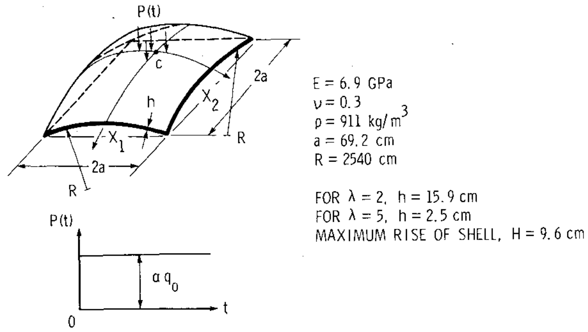


Fig. 4 Clamped shallow spherical shell segment and loading used.

Results obtained using the reduced system of equations are compared with those of the full system of equations in Fig. 3. The results indicate that the present reduction method predicts the nonlinear transient response accurately. Agreement is shown for the center vertical displacement in Fig. 3. The set of reduced basis vectors included eight steady-state vibration mode shapes and the nonlinear static solution at a load level corresponding to the intensity of the step load. The same set of reduced basis vectors was used throughout the analysis of the reduced system of equations. The number of unknown degrees of freedom is reduced by a factor of 20, and the comutational cost, based on the charging algorithm for the NASA Langley Research Center Computer Complex, of predicting the nonlinear dynamic response, is reduced by a factor of nearly 5. Also shown in Fig. 3 are results obtained by using an equivalent number of initial vibration mode shapes only. The set of initial vibration mode shapes leads to substantial error in the nonlinear transient response. This error results from the inability to represent the membrane-bending coupling effects introduced by the geometric nonlinearities.

Clamped Shallow Spherical Shell Segment with $\lambda = 2$

The behavior of the clamped shallow spherical shell segment with $\lambda = 2$ shown in Fig. 4 has been analyzed for the case of a uniformly distributed lateral load. Only one-quarter of the rectangular planform of the spherical shell segment was modeled using four 9-node mixed elements. A total of 128 stress-resultant degrees of freedom are nonzero. The external pressure is normalized by the classical buckling pressure q_0 of a complete spherical shell of the same radius of curvature and thickness,

$$q_0 = 32EH^3h/\lambda^2a^4 \tag{14}$$

where E is the elastic modulus and a is the half-length of a side of the shell.

The dynamic response of the clamped shallow spherical shell segment with $\lambda = 2$ to a step load intensity of $p = 22q_0$ was also determined. The linear and nonlinear transient responses of the shell predicted by the full system of equations are shown in Fig. 5. The addition of the nonlinear effects not only makes the shell stiffer but also decreases the response period as shown by the temporal variation of the center vertical displacement in Fig. 5a. Similarly, the ratio of extensional to bending strain energies from the nonlinear solution shown in Fig. 5b increases markedly near peaks of the center vertical displacement.

Results obtained using the reduced system of equations are compared with those of the full system of equations in Fig. 6. These results indicate that the present reduction method accurately predicts the nonlinear dynamic response of the shell. The set of reduced basis vectors of the present reduction method included 10 steady-state vibration mode shapes and the nonlinear static solution at a load level corresponding to the intensity of the step load. This set of reduced basis vectors was used throughout the analysis of the reduced system of equations. The number of unknown degrees of freedom is reduced by a factor of nearly 6 and the computational cost of predicting the nonlinear dynamic response is reduced by a factor of nearly 7. Also shown in Fig. 6 are results obtained by using an equivalent number of initial vibration mode shapes only. This set of reduced basis vectors again leads to substantial error in the nonlinear transient response, even though some membrane-bending coupling is present in these vibration mode shapes.

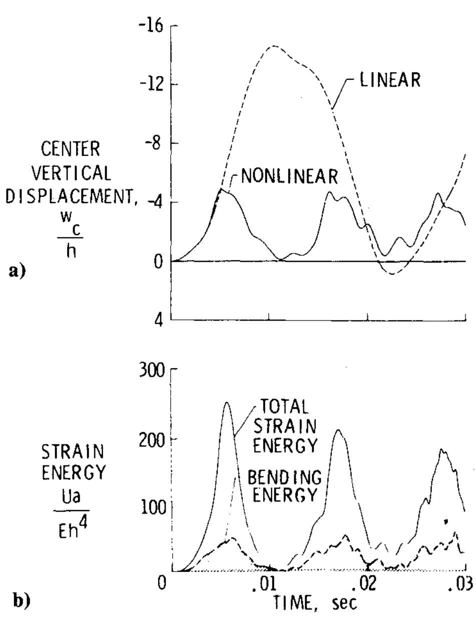


Fig. 5 Full system dynamic response of a clamped shallow spherical shell segment with $\lambda = 2$ subjected to a step external pressure load of $22q_0$ ($\Delta t = 50. \times 10^{-6}$ s).

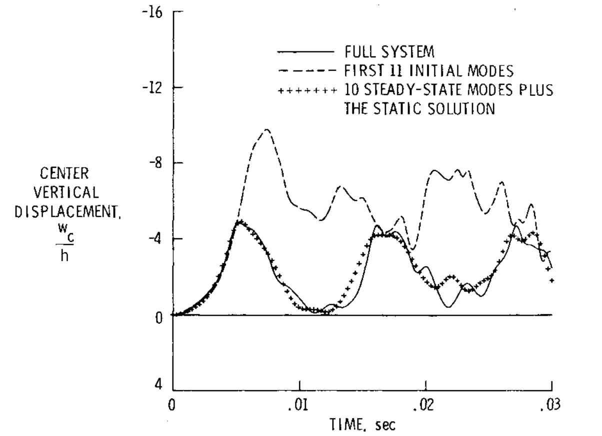


Fig. 6 Reduced system nonlinear dynamic response of a clamped shallow spherical shell segment with $\lambda = 2$ subjected to a step external pressure load of $22q_0$ ($\Delta t = 500. \times 10^{-6}$ s).

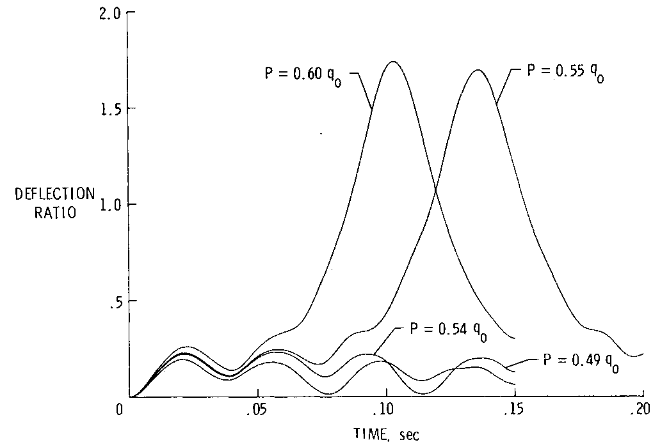
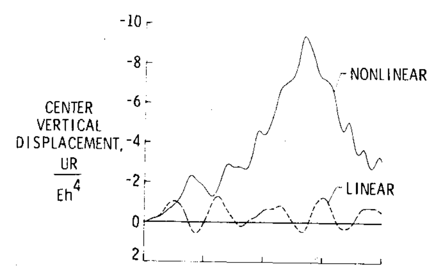
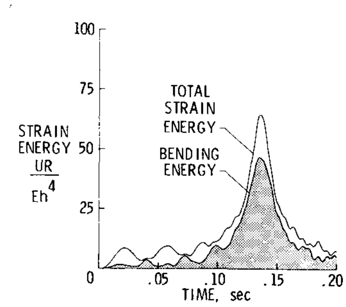


Fig. 7 Full system nonlinear dynamic response of a clamped shallow spherical shell segment with $\lambda = 5$ for increasing intensities of the external pressure.



a) Center vertical displacement.



b) Strain energy.

Fig. 8 Full system dynamic response of a clamped shallow spherical shell segment with $\lambda = 5$ subjected to a step external pressure load of $0.55q_0$ ($\Delta t = 50. \times 10^{-6}$ s).

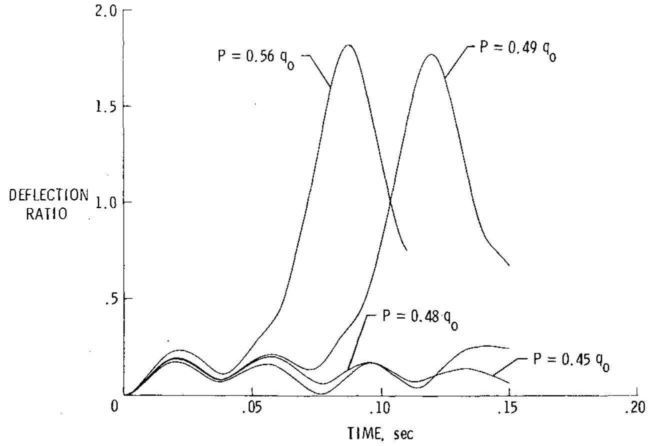


Fig. 9 Reduced system nonlinear dynamic response of a clamped shallow spherical shell segment with $\lambda = 5$ for increasing intensities of the external pressure.

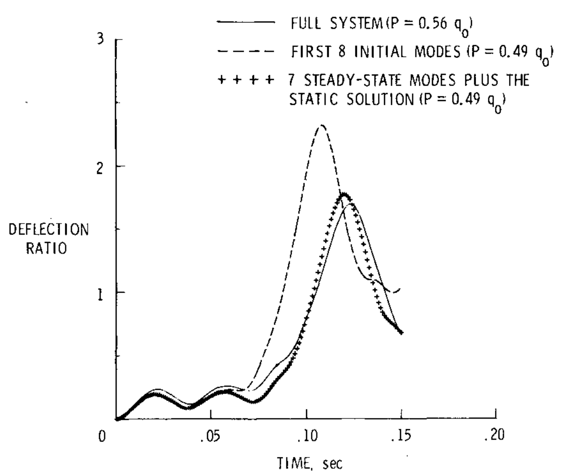


Fig. 10 Reduced system nonlinear dynamic response of a clamped shallow spherical shell segment with $\lambda = 5$ subjected to a step external pressure load that is slightly above the critical dynamic buckling load ($\Delta t = 500. \times 10^{-6}$ s).

Clamped Shallow Spherical Shell Segment with $\lambda = 5$

The behavior of the clamped shallow spherical shell segment with $\lambda = 5$ shown in Fig. 4 has been analyzed for the case of a uniformly distributed external pressure. The finite element model was identical to the finite element model of the clamped shallow spherical shell segment with $\lambda = 2$ except for the shell thickness. The nonlinear static response exhibits a limit point and two horizontal tangents. As the limit point is approached, the equilibrium configuration of the shell becomes increasingly sensitive to changes in the applied external pressure.

The dynamic response of the clamped shallow spherical shell segment with $\lambda = 5$ to a step pressure load and the symmetric dynamic buckling load were determined. The full system nonlinear dynamic responses of shell to increasing intensities of the external pressure are shown in Fig. 7. Dynamic instability is predicted to occur for an external pressure intensity lying between $p = 0.54q_0$ and $p = 0.55q_0$. The dynamic buckling load predicted by the two-dimensional finite element model is 4% higher than that predicted by Budiansky and Roth[14] for an axisymmetric analysis. The sensitivity of the structure to slight changes (less than 2%) in the applied load is shown in Fig. 7. The maximum value of the deflection ratio (ratio of the average vertical displacement to the average rise of the shell) increases only slightly and occurs earlier in time as the pressure intensity is increased above the value predicted for dynamic buckling. The linear and nonlinear transient responses of the shell are shown in Fig. 8, as predicted by the full system for an applied pressure load slightly above the dynamic buckling load. The nonlinear response shown in Fig. 8 indicates that the shell, in addition to being much more flexible than the linear solution predicts as shown in Fig. 8a, exhibits a change in equilibrium position compared to that of the linear solution. Also, dynamic buckling corresponds to a large increase in extensional energy over a relatively short period of time, as shown in Fig. 8b.

The dynamic buckling load of the shell was also determined using the present reduction method. The reduced system nonlinear dynamic responses of the shell to increasing intensities of the external pressure are shown in Fig. 9. These results were obtained using a single set of reduced basis vectors. The reduced basis vectors included seven steady-state vibration mode shapes and the nonlinear static solution at a pressure intensity of $p = 0.6q_0$, which is below the limit point value. The reduced system of equations generated using this set of reduced basis vectors is used throughout the dynamic analysis and for several intensities of the step load pressure in order to predict the dynamic buckling load. The number of unknown degrees of freedom is reduced by a factor of 8, and the computational cost of predicting the nonlinear dynamic response for each pressure intensity is reduced by a factor of 7. Considerable computational savings were achieved using the reduced system of equations since the free vibration eigenvalue analysis was performed only once.

Dynamic instability of the shell as predicted by the present reduction method occurs at an external pressure intensity lying between $p = 0.48q_0$ and $p = 0.49q_0$. This dynamic buckling load agrees with the axisymmetric dynamic buckling load predicted by other investigators.[14,15] However, the critical value predicted by the reduced system of equations is 13% below the critical value predicted by the full system of equations. The difference between the dynamic buckling loads of the full and reduced systems of equations is attributed to an inadequate modal representation of the spatial distribution of the lateral load. The class of structures having a softening behavior is sensitive to small changes in the applied load. The use of a truncated modal expansion and a given spatial discretization for which only a limited number of vibration mode shapes can be represented accurately are found to combine in such a way to amplify the sensitivity to variations in the applied load. The class of structures having a stiffening behavior is relatively insensitive to small changes in the applied load, and this error results in only slight changes in the amplitude and period of vibration.

The nonlinear dynamic response of the shell subjected to the dynamic buckling load is shown in Fig. 10. Using the full system of equations, results were obtained for a pressure intensity of $p = 0.56q_0$. Using the reduced system of equations, results were obtained for pressure intensity of $p = 0.49q_0$. The different values of pressure intensity for the analysis of the full system of equations and the analysis of the reduced system of equations were used to account for the inaccuracies introduced by the modal representation of the spatial distribution of the applied load. The results obtained using the present reduction method are comparable to those predicted using the full system of equations as shown in Fig. 10 for the deflection ratio. Also shown in Fig. 10 are results obtained using an equivalent number of initial vibration mode shapes only. This set of reduced basis vectors leads to an inaccurate transient response prediction.

Concluding Remarks

A modified modal method for the dynamic analysis of geometrically nonlinear structures subjected to step load has been presented. The method uses a combination of the nonlinear static solution for the applied loading under consideration and some selected vibration mode shapes. The vibration mode shapes correspond to the eigenvectors obtained by solving a standard free vibration eigenvalue problem in which the stiffness matrix is expanded about the nonlinear static solution. By including the static solution as a reduced basis vector, the time-averaged steady-state portion of the transient response may be readily characterized. This set of reduced basis vectors predicts the nonlinear dynamic response of plates and shells subjected to step loads better than a modal superposition approach using only the initial free vibration mode shapes.

A strategy has also been outlined for determining which and how many vibration modes shapes to include as reduced basis vectors. The strategy involves two criteria. The first criterion is used to specify the number of vibration mode shapes to calculate, while the second criterion is used to select which of the calculated mode shapes to include as reduced basis vector. The second criterion is based on the orthogonality of the spatial distribution of the applied load and the vibration mode shapes. This criterion provides a measure of the participation of each mode shape in the total solution.

The effect of inaccurate representation of the spatial distribution of the applied load has been identified for each class of structures considered. These inaccuracies are introduced by two mechanisms. The first mechanism results from using a truncated modal expansion. The second mechanism results from the use of a given spatial discretization of the structure for which only a limited number of vibration mode shapes can be represented accurately. The class of structures with a stiffening behavior are found to be nearly insensitive to the errors introduced in representing the applied lateral load; and this error results in only a slight change in the amplitude and period of vibration. Conversely, structures with a softening behavior that exhibit a limit point in the postbuckling response are very sensitive to small variations in the applied lateral load, regardless of whether the variation results from a change in the load intensity or from errors introduced by the modal representation of the spatial distribution of the applied load. To include in the set of reduced basis vectors the vibration mode shapes corresponding to higher vibration frequencies requires the use of a sufficiently fine finite element mesh to represent these more complex vibration mode shapes. The finite element mesh required to adequately approximate higher vibration mode shapes may involve more elements than the mesh required to obtain a converged nonlinear dynamic response using the full system of equations. The computation cost of obtaining the reduced basis vectors will be increased, but the computation cost of solving the reduced system of equations will still be much less than that of predicting the nonlinear dynamic response using the full system of equations.

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